Midterm take-home exam

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Directions

- Use A4 or US Letter paper. If you need more than one piece of paper, bind the paper by a staple.
- Typing with a computer is desirable. If you handwrite, use pens.
- If you find my mistakes in a problem, discuss and correct them and then write your solution. If you believe that fewer assumptions are given than necessary, discuss why you believe so, add assumptions you think are necessary and then write your solution.
- In some problems, you are asked to draw graphs. Rough sketches suffice but you'll get a bonus of several points for program-generated graphs. **Don't forget to include the codes**. (I would recommend you to use R Markdown/ R Notebook on RStudio, which can be exported to a PDF file and MS Word document.)
- You can work with other students. For academic integrity, append acknowledgement section to list all students who helped you (not who you helped). And yet, be responsible for your solution.

[1] Standard Solow model

Consider the standard Solow model. Assume that the economy is on the balanced growth path at $t = 0$.

(1) Suppose that there is a permanent increase in both L and n due to a change of immigration policy at $t = t_0 > 0$. What are the short-run and long-run effects on the economy? Draw graphs and discuss.

(2) Suppose that there is a parmanent increase in g probably due to a change of science, education and/or patent policies at $t = t_0 > 0$. What are the short-run and long-run effects on the economy? Draw graphs and discuss.

[2] Solow model with government purchases

Consider a Solow economy with government purchases under otherwise standard assumptions. In particular, assume that the goverment buys output at rate $G(t)$ per unit of effective labor per unit time. The capital accumulation equation becomes

$$\dot{k}(t) = sf(k(t)) - G(t) - (\delta + g + n)k(t).$$

Suppose that the economy is on the balanced growth path at $t = 0$ with $G(t) = G_0 > 0$ for $t \leq 0$.

(1) Suppose that there is a permanent increase in G at $t = t_0 > 0$, that is, for some $G_1 > G_0$ it holds that

$$G(t) = \begin{cases} G_0 & \text{for } 0 \leq t < t_0 \\ G_1 & \text{for } t \geq t_0 \end{cases}$$

What are the short-run and long-run effects on the economy? Draw graphs and discuss.

(2) Suppose that there is a temporary increase in G at $t = t_0 > 0$, which ends that $t = t_1$. For some $G_1 > G_0$ it holds that

$$G(t) = \begin{cases} G_0 & \text{for } 0 \leq t < t_0 \\ G_1 & \text{for } t_0 \leq t \leq t_1 \\ G_0 & \text{for } t \geq t_1 \end{cases}$$

What are the short-run and long-run effects on the economy? Draw graphs and discuss.

[3] Solow model with capital-augmenting technical progress

Consider a Solow economy with capital-augmenting technical progress under otherwise standard assumptions. In particular, assume

$$Y = F(AK, L) = (AK)^{\alpha} L^{1-\alpha}, \quad \dot{A}(t) = \mu A(t), \quad t \geq 0.$$

(1) Let $k = \frac{K}{A^{\phi} L}$, $\phi = \frac{\alpha}{1-\alpha}$ and compute the intensive form production function $y = f(k) = F(k, 1)$.

(2) Show that the k converges to a steady state k^*.

(3) Find the growth rates of Y, K, Y/L, and K/L when $k = k^*$.

(4) Draw graphs of Y, K, Y/L and K/L as functions of time. Use log scales for the vertical axes.

(5) **Optional bonus problem**. The above results are based on the assumption that the production function is Cobb–Douglas. Is it possible to extend the above analysis to the case of a general CRS production function? Experiment and discuss the (im)possibility.

[4] CES production functions

CES (Constant Elasticity of Substitution) functions are a family of functions of the form:

$$F(K, L) = [\alpha K^{\gamma} + (1-\alpha) L^{\gamma}]^{\frac{1}{\gamma}}, \quad 0 < \alpha < 1, \quad \gamma < 1.$$

(a) Show that F has constant returns to scale.

(b) Show that the margianl rate of technical substitution, defined by$MRTS_{KL} := \left(\frac{\partial F}{\partial L}\right) / \left(\frac{\partial F}{\partial K}\right)$, satisfies

$$MRTS_{KL} = \frac{\alpha}{1-\alpha} \left(\frac{K}{L}\right)^{1-\gamma}.$$

(c) Represent $\ln\left(\frac{K}{L}\right)$ as a function of $\ln MRTS_{KL}$ to show that the elasticity of substitution, defined by,

$$\sigma = \frac{d \ln (K/L)}{d \ln MRTS_{KL}},$$

satisfies

$$\sigma = \frac{1}{1-\gamma}.$$

(d) The Cobb–Douglas family of production functions constitutes a subset of CES functions. Verify this fact by taking the limit of $\gamma \to 0$. [Hint: Use l'Hôpital's rule to compute $\lim_{\gamma \to 0} \ln F(K, L)$.]

[5] Utility functions

As a choice of utility function, CRRA (Constant Relative Risk Aversion) functions are often used in macroeconomics. The following functions are instances of CRRA functions:

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0, \\ \ln c & \text{if } \theta\text{=1.} \end{cases}$$

(a) The relative risk aversion is defined by

$$-\frac{cu''(c)}{u'(c)}.$$

Show that the CRRA functions have constant relative risk aversion. More specifically,

$$-\frac{cu''(c)}{u'(c)} = \theta, \qquad \text{for all } c > 0.$$